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Instability of N-body systems: can computers solve the problem?

V G Gurzadyan† and A A Kocharyan‡

† International Center for Relativistic Astrophysics, Dipartimento di Fisica, Universita di Roma 'La Sapienza', 00185, Rome, Italy
and Department of Theoretical Physics, Yerevan Physics Institute, Yerevan 375036, Armenia
‡ Department of Mathematics, Monash University, Clayton, Victoria 3168, Australia

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Abstract. The problem of the numerical study of instability of N-body gravitating systems by means of Lyapunov characteristic exponents is considered. The discontinuity of Lyapunov exponents is shown for computer-created systems both with softened and, what is more interesting, unsoftened (i.e. pure Newtonian) potentials. The Lyapunov technique thus cannot be considered as an appropriate method of study of N-body systems, and physical and astrophysical interpretations of results of previous computer studies appear to be unfounded.

The numerical study of instability of N-body gravitating systems has, since the pioneering paper by [1], become one of the important areas of computer simulations. It is partly determined by the relation of instability properties of that system with relaxation-driving mechanisms in star clusters and galaxies. The latter factor became crucial after the proof of exponential instability of spherical N-body systems [2, 3] and the evidence of essential consequences it can have for stellar dynamics.

Numerical studies performed to investigate this problem are based on the calculation of the growth of perturbations by time (see [4, 5] and references therein), considered to be Lyapunov characteristic exponents of the system.

In the present paper, therefore, we investigate the problem of validity of the Lyapunov exponent technique to the computer study of N-body problems. We show that the calculations of Lyapunov exponents for N-body systems with softened and even unsoftened (!) potential can have no any relation to properties of the corresponding real system.

We approach this problem from the concept of the theory of dynamical systems, enabling us to arrive at general conclusions, valid for d-dimensional Hamiltonian systems with any potential.

Consider a smooth d-dimensional manifold M, with a given σ-field of measurable sets $\mathcal{B}(M)$ and complete measure P ($P(M) = 1$).

Let $\{f^t\}$ be a group of diffeomorphisms on M with continuous $t \in R$ (or discrete $t \in Z$) time:

$$f^t : M \to M \qquad f^0 = \mathrm{id} \qquad f^{s+t} = f^s \circ f^t$$

for arbitrary times t, s. Then $(M, \mathcal{B}(M), P, f)$ we will call a dynamical system with continuous (or discrete) time. The dynamical system is measure-preserving if

$$P(f^t A) = P(A) \qquad \forall A \in \mathcal{B}(M) \qquad \forall t \in R\,.$$

We denote by $\mathcal{D}$ the space of all dynamical systems $(M, \mathcal{B}(M), P, f)$ with appropriate topology.

We define a function Φ on $\mathcal{D}$:

$$\Phi : \mathcal{D} \to R\,.$$

Well known examples of such functions are Kolmogorov–Sinai (KS) entropy and Lyapunov mean-characteristic exponents:

$$\chi_i = \int_M \chi_i(x) P(\mathrm{d}x) \qquad i = 1, \ldots, q$$

where $\chi_i(x)$ are Lyapunov characteristic exponents

$$\chi(x) = \lim_{t\to\infty} \frac{\ln \|\mathrm{d}f^t x\|}{t}$$

and

$$\chi_1 > \chi_2 > \cdots > \chi_q\,.$$

Now we argue the crucial role of the following question: *is Φ a continuous function?*

At first sight the answer to this question does not seem to be of much interest, since when one deals with a given dynamical system the behaviour of a certain function for different systems is not important.

It is undoubtedly so for analytical methods, while for computer studies this point can become of extreme importance. Let us illustrate this idea by the following example.

Consider the manifold

$$M = S^1 = R^1/Z = \{\theta | 0 \leqslant \theta < 1\}$$

with given Lebesgue measure $P(S^1) = 1$, and class of dynamical systems

$$f_\alpha : S^1 \to S^1 : \theta \mapsto \{\theta + \alpha\}$$

where the brackets denote a fractional part.

Define Φ in the following way:

$$\Phi(f_\alpha) \equiv \Phi(\alpha) = \begin{cases} 1 & f_\alpha \text{ is ergodic} \\ 0 & f_\alpha \text{ is not ergodic}\,. \end{cases}$$

Evidently f_α is ergodic when α is an irrational number, and is non-ergodic when α is rational, i.e.

$$\Phi(\alpha) = \begin{cases} 1 & \alpha \text{ is irrational} \\ 0 & \alpha \text{ is rational}\,. \end{cases}$$

Now if one tries to evaluate $\Phi(\sqrt{2})$, one will find out whether the dynamical system $f_{\sqrt{2}}$ is ergodic or not, say, via looking for the periodicity of orbits. Since the computer cannot deal with irrational numbers one is forced to study the following sequence of dynamical systems $\{f_{\alpha_n}\}$, where α_n's are rational numbers for any n and

$$\lim_{n\to\infty} \alpha_n = \sqrt{2}\,.$$

One may expect the following result

$$\Phi(\sqrt{2}) = \lim_{n\to\infty} \Phi(\alpha_n) = 0$$

though the real value of $\Phi(\sqrt{2})$ is 1. So

$$\text{Computer image of } f_{\sqrt{2}} \neq \text{ Real image of } f_{\sqrt{2}}.$$

Therefore one can never figure out by computer methods $\Phi(\alpha)$ when α is irrational. This example clearly shows what in fact happens when one tries to study a dynamical system by computer.

In a typical case one is forced to consider not the diffeomorphisms f^t but

$$f_\varepsilon^t = f^t + \varepsilon(t)$$

because of the inevitable computing errors $\varepsilon(t)$ arising at truncation of numbers. As a result, when Φ is not a continuous function all numerical calculations can completely lose their meaning, i.e. the computer's $\Phi^c(f)$ does not coincide with real one, $\Phi(f)$.

In some cases such a situation can arise as a result of deliberate change of the behaviour of f to avoid 'naughty' functions, as it just happens at 'softening' of the Newtonian potential.

Thus we arrive at the following two questions reflecting both fundamental properties of function Φ:

(S) Is $\Phi(f_\varepsilon)$ close to its computer image $\Phi^c(f_\varepsilon)$? (stability);
(C) Is $\Phi(f)$ the limit of $\Phi(f_\varepsilon)$ as ε (bifurcation parameter) tends to zero? (continuity).

As a representation of particular interest of Φ we shall consider Lyapunov characteristic exponents.

First, recall the following quite remarkable results.

1(a) According to Mañé's theorem (see [6]) when M is a compact surface, C^1 area-preserving non-Anosov diffeomorphisms, all of whose Lyapunov exponents are equal to zero Lebesgue almost everywhere, and are everywhere dense.
1(b) In general, Lyapunov exponents are discontinuous functions of a bifurcation parameter [7].
1(c) Topological entropy is proved to be discontinuous for $\dim M \geqslant 4$ [8].

We see that Lyapunov exponents can be highly discontinuous.

The example considered above can be an illustration of this typical property of function Φ.

On the other hand one has the following properties.

2(a) Topological entropy is continuous at $\dim M = 2$ [8].
2(b) For some dynamical systems it is proved that Lyapunov exponents are upper-continuous [9].
2(c) Though typically Lyapunov exponents are highly discontinuous, there exists regular family of perturbations fulfilling conditions discussed in [6] *making them stable.*

We see that while the properties 1(a)–(c) make doubtful the usefulness for computations of Lyapunov exponents, properties 2(a)–(c) leave some hope. What is evident is the necessity of thorough consideration of this problem in any given particular case.

In view of the results described above, let us turn to our problem of the stability in the case of an N-body system.

First remember that the trajectories of Hamiltonian system

$$H(p,q) = \tfrac{1}{2} g^{\mu\nu} p_\mu p_\nu + V(q)$$

in the region of configurational space

$$M = \{q | V(q) < E\}$$

can be represented as geodesics of M with a Riemannian metric

$$G = [E - V(q)]g \equiv Wg$$

and affine parameter s:

$$\mathrm{d}s = \sqrt{2}W\,\mathrm{d}t\,.$$

It is also well known that the stability properties of geodesics on unit vector bundle

$$SM = \{u \in TM \mid G(u, u) = 1\}$$

can be determined by the behaviour of Riemann ($Riem$), Ricci (Ric) and scalar (R) curvatures.

Indeed, Jacobi equation

$$\nabla_u \nabla_u n + Riem(n, u)u = 0$$

where ∇ is the Levi–Civita connection of the metric G, and vectors u and n denote the velocity on geodesics and their deviation, shows that behaviour of deviation vector n depends on Riemannian tensor $Riem$.

Moreover, if M is a compact manifold with local symmetric metric G, and negative sectional curvature, then one may obtain KS-entropy $h(f)$ of geodesical flow using either the following formula:

$$h(f) = \int_{SM} \mathrm{tr}(\sqrt{-\Re_u})\,\mathrm{d}\mu$$

where

$$\Re_u v = Riem(v, u)u$$

or Pesin's formula

$$h(f) = \int_{SM} \sum_{\chi(u)>0} \chi(u)\,\mathrm{d}\mu$$

where χ are the Lyapunov exponents.

Comparing these two formulas, one can readily infer that

$$\chi(u)^2(\delta - u \otimes u) \sim -\Re_u\,.$$

For local symmetric metrics, as one has, for example, while looking for the motion of photon beam in Friedmann Universe with negative curvature [10], it reads

$$\chi(u)^2 \sim -\frac{R}{d(d-1)}\,.$$

So, if one averages in space and time reparametrization, then for a general dynamical system one arrives at

Instability mean index $\sim$ Lyapunov exponent.

Let us recall that in order to obtain Lyapunov exponents one has to integrate a dynamical system and linearize it for t which tends to infinity. Of course it is an impossible task, if one evaluates Lyapunov exponents by means of computer simulations. Therefore one is forced to consider only finite parts of trajectories of the dynamical system. Besides, as far as the N-body gravitating system is concerned, one comes across singularities. So the instability mean index is just connected with those Lyapunov exponents which are obtained in computer calculations.

Therefore, the problem of continuity of Lyapunov exponents is reduced to the study of the continuity of the Riemannian tensor, particularly since the latter is not continuous if the scalar curvature R is discontinuous. So the problem amounts to the investigation of the *instability mean index* λ^2—a measure of average (in space) instability for the d-dimensional Hamiltonian system:

$$\lambda^2 \equiv -\frac{2RW^2}{d(d-1)} = \frac{2\Delta W}{d} + \left(\frac{1}{2} - \frac{3}{d}\right)\frac{\|\mathrm{d}W\|^2}{W}$$

where the time reparametrization is made and

$$\|\mathrm{d}W\|^2 = g^{\mu\nu}\frac{\partial W}{\partial q^\mu}\frac{\partial W}{\partial q^\nu} \qquad \Delta W = g^{\mu\nu}\frac{\partial^2 W}{\partial q^\mu \partial q^\nu}\,.$$

Previously, in [11], we introduced a measure of relative instability based on the value of Ricci curvature.

In the case of N-body gravitating system one has $d = 3N$ and

$$V(q) = -\sum_{a=1}^{N}\sum_{b=1}^{a-1} GM_aM_b\varphi(r_{ab}) \qquad r_{ab}^2 = (r_{ab}^1)^2 + (r_{ab}^2)^2 + (r_{ab}^3)^2 \qquad r_{ab}^i = r_a^i - r_b^i$$

where the function φ is not specified yet. Then one has

$$g_{\mu\nu} = M_\mu\delta_{\mu\nu} \qquad \mu = (a, i) \qquad \delta_{\mu\nu} = \delta_{ab}\delta_{ij} \qquad M_\mu = M_a \qquad q^\mu \equiv r_a^i$$

where $a = 1, \ldots, N$ and $i = 1, \ldots, 3$.

Calculating the instability mean index, taking into account that

$$W = \sum_{a=1}^{N}\frac{|p_a|^2}{2M_a}$$

$$\|\mathrm{d}W\|^2 = \sum_{a=1}^{N} G^2M_a\sum_{i=1}^{3}\left(\sum_{c=1,c\neq a}^{N} M_c\varphi'(r_{ac})\frac{r_{ac}^i}{r_{ac}}\right)^2 = \sum_{a=1}^{N}\frac{|F_a|^2}{M_a}$$

$$\Delta W = \sum_{a=1}^{N}\sum_{c=1,c\neq a}^{N} GM_cr_{ac}^{-2}(r_{ac}^2\varphi'(r_{ac}))'$$

one has

$$\lambda^2 = \Lambda_1 + \Lambda_2$$

$$\Lambda_1 = \frac{2}{3N}\sum_{a=1}^{N}\sum_{c=1,c\neq a}^{N} GM_cr_{ac}^{-2}(r_{ac}^2\varphi'(r_{ac}))'$$

$$\Lambda_2 = \left(\frac{1}{2} - \frac{1}{N}\right)\sum_{a=1}^{N}\frac{|F_a|^2}{M_a}\Bigg/\sum_{a=1}^{N}\frac{|p_a|^2}{2M_a}$$

where the following notations are used:

$$|F_a|^2 = (F_a^1)^2 + (F_a^2)^2 + (F_a^3)^2$$

$$F_a^i = \sum_{c=1,c\neq a}^{N} F_{ac}^i = \sum_{c=1,c\neq a}^{N} GM_aM_c\varphi'(r_{ac})\frac{r_{ac}^i}{r_{ac}}\,.$$

Now consider a class of potentials $\varphi_\varepsilon(r)$ containing the two main cases.

(i) Newtonian potential ($\varepsilon = 0$):

$$\varphi_0(r) = \frac{1}{r}\,.$$

(ii) Softened Newtonian potential ($\varepsilon \neq 0$):

$$\varphi_\varepsilon(r) = \frac{1}{\sqrt{r^2 + \varepsilon^2}}\,.$$

Let us look for the behaviour of λ when both r and ε are close to zero, i.e. for the continuity of the mean index in the physically most interesting case. For this purpose one has to obtain the following limits:

$$\lim_{\varepsilon\to 0}\lim_{r\to 0} \lambda^2(r, \varepsilon) = -\infty$$

$$\lim_{r\to 0}\lim_{\varepsilon\to 0} \lambda^2(r, \varepsilon) = +\infty\,.$$

When $\varepsilon = 0$, i.e. in the case of an unsoftened potential, the mean index is determined by Λ_2 and the system is exponentially unstable, since

$$\lambda^2 \sim r^{-3} \qquad \text{as} \quad r \to 0\,.$$

This limit corresponds to the close encounter of at least two particles.

The same limit when $\varepsilon \neq 0$, for the softened potential, reveals completely different behaviour: the mean index is a complex number,

$$\lambda^2 \sim -\varepsilon^{-3} \qquad \text{as} \quad \varepsilon \to 0$$

since it is determined by first member Λ_1; as a result the system is not unstable any more.

We see that the mean instability index and hence Lyapunov exponents in the case of an unsoftened potential are discontinuous

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon \neq \lambda_0\,.$$

Moreover, the unsoftened system has quite different behaviour, particularly in accord with point 2(b) of section 2, and is more stable than the original Newtonian system.

A qualitatively similar situation is in the case of another perturbed potential

$$\varphi_\varepsilon(r) = \frac{1}{r + \varepsilon}\,.$$

A marked dependence of the growth of initial errors on the parameter of softening has already been noticed during computer simulations [12, 13].

Thus the calculations of Lyapunov exponents by means of computer methods for N-body systems cannot lead to any meaningful results.

Already this fact is enough to seriously influence the conclusions of numerous computer studies of the instability of softened systems (see [4]). Other difficulties of those studies, particularly concerning the interpretation of relaxation-type effects, were outlined in [14].

However, the next conclusion of the present study is even more radical: the principal impossibility of the investigation of the instability of not only disturbed but even $1/r$-potential N-body systems by computers.

These conclusions demonstrate the necessity for the creation of new computer codes to describe the N-body system with phase trajectory close to the physical one for long enough timescales (in a physical sense).

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